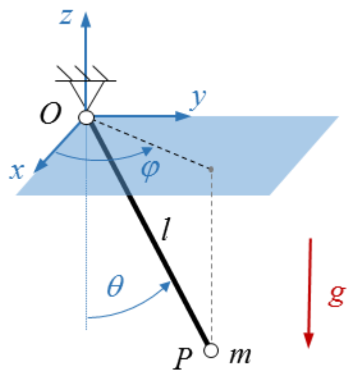
LINEARIZATION OF THE DYNAMICS OF A SIMPLE 3D PENDULUM

Figure 1 shows a simple 3D pendulum. The system is subjected to gravity effects (9.81 m/s^2 along the negative direction of the *y*-axis). Rod *O-P* is massless; all the system mass is located at point *P*.



Physical properties

Rod length, *l*: 1 m
Point mass, *m*: 1 kg

Figure 1: A simple 3D pendulum

The system has two degrees of freedom; its dynamics can be described by means of two ODEs in terms of two angular coordinates: θ, which represents the angle between rod *O-P* and the negative *z* axis, and the angle φ from the *x* axis to the projection of rod *O-P* onto the horizontal *xy* plane:

$$l\,\ddot{\theta} - l\sin\theta\cos\theta\ \dot{\varphi}^2 + mgl\sin\theta = 0$$
$$ml^2\left(\ddot{\varphi}\sin^2\theta + 2\,\dot{\varphi}\dot{\theta}\sin\theta\cos\theta\right) = 0$$

The system is in static equilibrium at an angle $\theta = 0$, for any value of φ. The linearized dynamics about this equilibrium configuration is given by

$$\mathbf{M}_e\delta\ddot{\mathbf{y}} + \mathbf{K}_e\delta\mathbf{y} = \mathbf{0}$$

where

$$\mathbf{M}_e = \begin{bmatrix} l & 0 \\ 0 & 0 \end{bmatrix}; \quad \mathbf{K}_e = \begin{bmatrix} g & 0 \\ 0 & 0 \end{bmatrix}; \quad \mathbf{y} = \begin{bmatrix} \theta \\ \varphi \end{bmatrix}$$

The linearized dynamics equations have a pair of purely imaginary eigenvalues $s_{1,2} = \pm i\sqrt{g/l}$, as well as a double eigenvalue that equals infinity, which stems from the fact that φ is a cyclical coordinate.

If the system is modelled using as variables the *x* and *y* coordinates of point *P* then the pendulum dynamics is expressed by a different system of ODEs, whose linearized expression is

$$\mathbf{M}_q\delta\ddot{\mathbf{q}} + \mathbf{K}_q\delta\mathbf{q} = \mathbf{0}$$

where

$$\mathbf{M}_q = \begin{bmatrix} l & 0 \\ 0 & l \end{bmatrix}; \quad \mathbf{K}_q = \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}; \quad \mathbf{q} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The system spectrum now consists of two identical pairs of imaginary eigenvalues, $s_{1,2} = s_{3,4} = \pm i\sqrt{g/l}$.

The two above-mentioned descriptions of the system dynamics convey the same information about the mechanical system, although in different formats. This highlights that, even in a simple problem like this one, the choice of generalized coordinates determines the way in which the linearization process delivers its results.

Multibody dynamics formulations can express the system dynamics either as a set of ODEs or a system of DAEs. Depending on the coordinate selection and the formulation properties, the linearized dynamics can be a linear system with the exact spectrum of the constrained system, or with an approximation of it. Spurious eigenvalues may also be obtained; these must be discriminated from the real system spectrum.

For benchmarking purposes, the accuracy of a linearization method in the solution of this problem is defined as the norm-2 of the array that contains the difference between the exact eigenvalues and the true eigenvalues predicted by the selected method. This array may have dimension 2×1 or 4×1, depending on the variables selected to describe the system. Given the small system size, the efficiency of the method is defined as the time elapsed in 10,000 executions of the linearization process, which includes the evaluation of the necessary linearization terms, the evaluation of the system eigenvalues, and the discrimination between true and spurious eigenvalues, if necessary. The solution file should include the eigenvalues yielded by the method, both the real and the spurious ones.