Uncontrolled bicycle

Figure 1 shows the multibody model of an uncontrolled bicycle composed of four rigid bodies: the rear frame assembly (which includes the rider body), the front frame assembly (fork and handlebar), rear wheel, and front wheel. The gravity is the only force applied to the model, with a value of 9.81 m/s^2 along z axis. All the parameters of this bicycle are taken from [1], where linearized model of this bicycle is also studied.

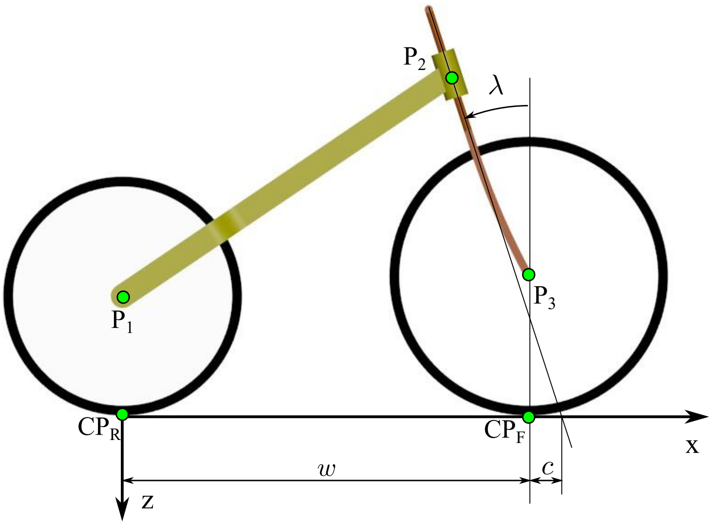

Figure 1: Bicycle model.

The x axis points forward, the y axis points towards the reader and the z axis points downwards. The coordinates of the points are given in Table 1. The coordinates of point P_2 are given for the sake of easiness, although it can be at any position as long as it belongs to the steering axis, as follows:

$$x_{P2} = w + c - k\sin(\lambda) \tag{1}$$
$$y_{P2} = 0 \tag{2}$$
$$z_{P2} = -k\cos\lambda \tag{3}$$

Table 1: Points of the bicycle model.

Point	Coordinates (x,y,z), in meters
P_1	$(0, 0, -0.3)$
P_2	$(0.82188470506, 0, -0.85595086466)$
P_3	$(1.020, 0, -0.35)$

The inertial properties and dimensions of the model are given in Table 2. The moments of inertia are referred to the center of mass, following the orientation of the global axes.

If perfect rolling contact is assumed in both wheels, this system has three degrees of freedom: the tilt angle (roll), the steering angle and the forward displacement of the whole bicycle.

A linearized model of this bicycle was studied in [1], and it was demonstrated that this bicycle is stable at forward speeds in between 4.29238253634111 and 6.02426201538837 m/s.

This model is a great benchmark problem for general purpose multibody simulation programs, since it is a conservative system and the range of stability of the bicycle is known from the stability analysis of the linearized model.

Table 2: Parameters for the benchmark bicycle shown in fig. 1 are described in the text. All the data is taken from [1]. The values given are exact (no round-off). The inertia components and angles are such that the principal inertias (eigenvalues of the inertia matrix) are also exactly described with only a few digits. The tangents of the angles that the inertia eigenvectors make with the global reference axes are rational fractions.

parameter	symbol	value for benchmark
wheel base	w	1.02 m
trail	c	0.08 m
steer axis tilt	λ	$\pi/10$ rad
gravity	g	9.81 N kg^{-1}
Rear wheel R		
radius	r_R	0.3 m
mass	m_R	2 kg
mass moments of inertia	(I_{Rxx}, I_{Ryy})	$(0.0603, 0.12)$ kg m^2
rear **B**ody and frame assembly B		
position center of mass	(x_B, z_B)	$(0.3, -0.9)$ m
mass	m_B	85 kg
mass moments of inertia	$\begin{bmatrix} I_{Bxx} & 0 & I_{Bxz} \\ 0 & I_{Byy} & 0 \\ I_{Bxz} & 0 & I_{Bzz} \end{bmatrix}$	$\begin{bmatrix} 9.2 & 0 & 2.4 \\ 0 & 11 & 0 \\ 2.4 & 0 & 2.8 \end{bmatrix}$ kg m^2
front **H**andlebar and fork assembly H		
position center of mass	(x_H, z_H)	$(0.9, -0.7)$ m
mass	m_H	4 kg
mass moments of inertia	$\begin{bmatrix} I_{Hxx} & 0 & I_{Hxz} \\ 0 & I_{Hyy} & 0 \\ I_{Hxz} & 0 & I_{Hzz} \end{bmatrix}$	$\begin{bmatrix} 0.05892 & 0 & -0.00756 \\ 0 & 0.06 & 0 \\ -0.00756 & 0 & 0.00708 \end{bmatrix}$
Front wheel F		
radius	r_F	0.35 m
mass	m_F	3 kg
mass moments of inertia	(I_{Fxx}, I_{Fyy})	$(0.1405, 0.28)$ kg m^2

Proposed maneuvers

Three maneuvers are proposed to be performed with this model. The starting position for all the tests is with the bicycle in vertical position and the steering straight. Then, it is launched with an initial forward velocity and some roll velocity as a perturbation. The first maneuver is below the range of stable speed. The second one is at the stable speed range, and the third one is over the stable speed range. The initial conditions are provided in table 3.

Results

The file of the results contains the most relevant data from 20 seconds simulations. Every row of the file is a time step. The first column of the results is the time (in seconds). Next 8 columns have data from maneuver 1: roll angle, roll angular velocity, forward speed, potential energy, kinetic energy, mechanical energy, steer angle, and steer velocity. Every 8 columns have the same data for maneuvers 2, and 3. All the units employed are from the International System of Units. The data have 14 significant digits.

Most representative data are shown in figs. 2 to 4 for maneuvers below the stable speed range, at the stable speed range, and over the speed range respectively.

The performance of the simulation can be described by measuring the percentage of variation

Table 3: Initial conditions for the tests.

Maneuver \ Initial velocity of DOFs	Forward speed (m/s)	Roll angle speed (rad/s)
Maneuver 1	4.0	0.05
Maneuver 2	4.6	0.50
Maneuver 3	8.0	0.05

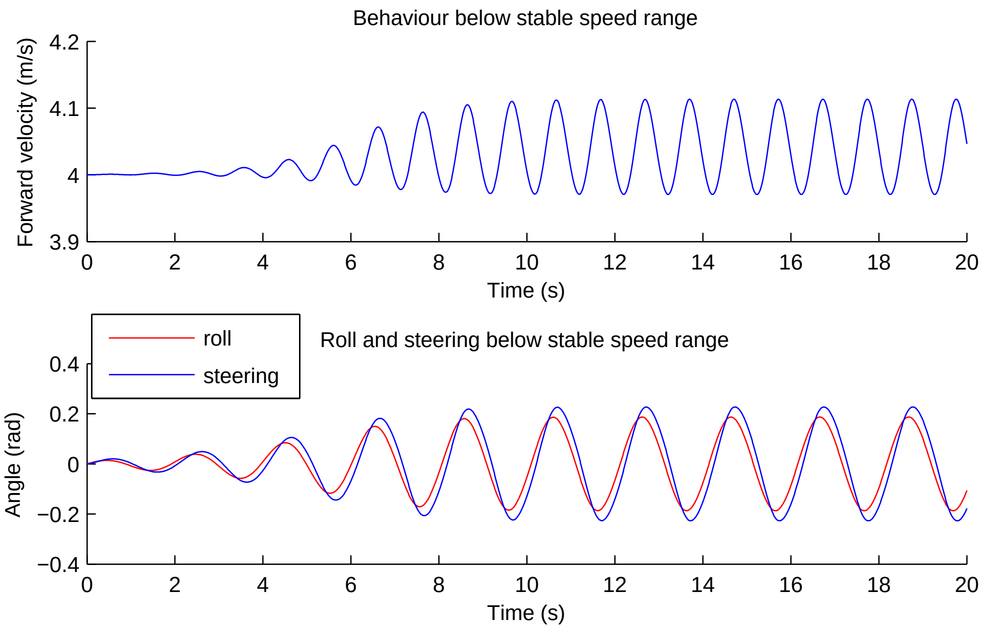

Figure 2: Maneuver 1. Below the stable speed range any perturbation becomes a big oscillation. If the speed is low enough, the bicycle ends up falling.

of mechanical energy. It can be calculated as follows:

$$performance = 100\frac{(\max(E_m) - \min(E_m))}{E_{m0}} \tag{4}$$

The objective of this benchmark is to carry out 20 seconds of simulation as fast as possible, while keeping the value of the performance in eq. (4) under 10^{-3} when applied to maneuver 2. The potential energy is calculated taking the floor as reference.

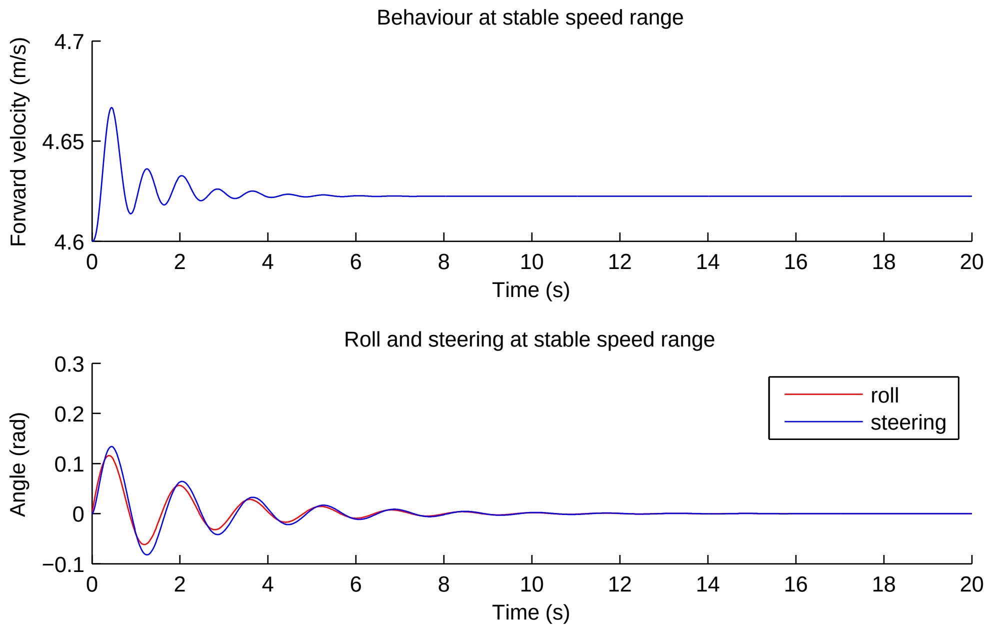

Figure 3: Maneuver 2. At any speed in the stable range, the amplitude of the oscillations diminishes in a seemingly damped way.

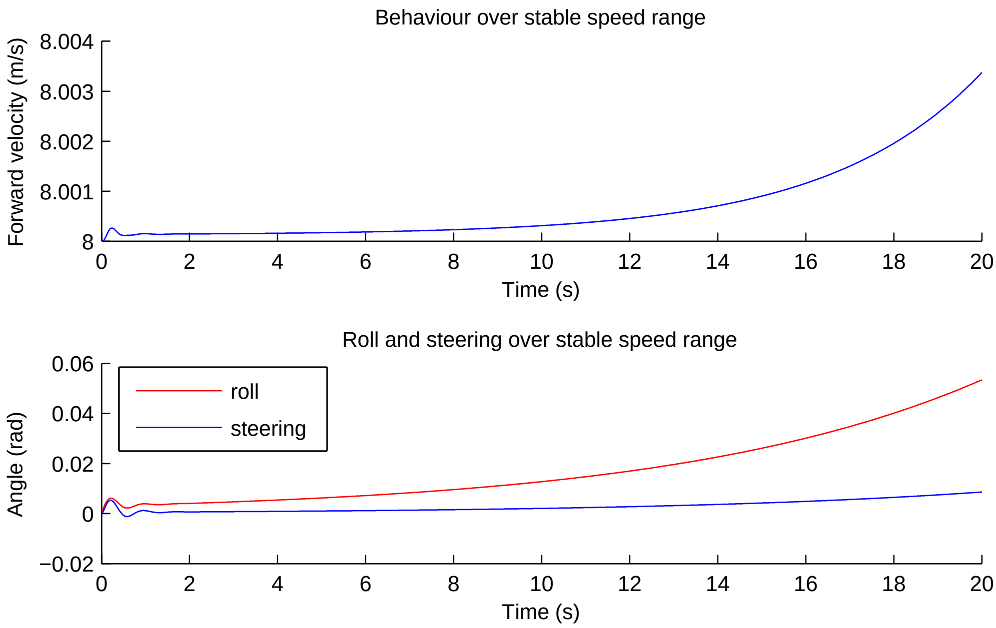

Figure 4: Maneuver 3. At speeds over the stable range, the trajectory of the bicycle is a spiral with decreasing radius. The bicycle increases its lean angle continuously until it falls.

Bibliography

[1] J.P. Meijaard, J.M. Papadopoulos, A. Ruina, and A.L. Schwab. Linearized dynamics equations for the balance and steer of a bicycle: a benchmark and review. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 463(2084):1955–1982, 2007.

Notes:

Useful definitions and equations are provided hereafter to calculate some of the magnitudes asked for in this problem.

The forward speed is defined as follows:

$$fs = \mathbf{v}_{P_1}^{\top} \frac{\mathbf{e_x} - \left(\mathbf{e_x}^{\top} [0\ 0\ 1]^{\top}\right) [0\ 0\ 1]^{\top}}{\left\| \mathbf{e_x} - \left(\mathbf{e_x}^{\top} [0\ 0\ 1]^{\top}\right) [0\ 0\ 1]^{\top} \right\|} \tag{5}$$

where $\mathbf{v}_{P_1}$ is the velocity of the point P_1, and $\mathbf{e_x}$ is an unit vector fixed to the rear frame, parallel to the global x axis at the beginning of the simulation. As this vector is not always parallel to the ground, its z coordinate must be removed, and the resulting vector normalized. Then, the velocity of point P_1 is projected over this vector.

The attitude of a vehicle can be expressed by means of aircraft principal axes, using the roll (ϕ), pitch (θ), and yaw (ψ) angles. In this example they are applied to the rear frame of the bicycle. The order of the rotations is yaw-pitch-roll if they are expressed as z-y-x intrinsic rotations, or roll-pitch-yaw if they are expressed as x-y-z extrinsic rotations.

A vector $\mathbf{u_b}$ expressed in the body reference system can be transformed into a vector $\mathbf{u_g}$ expressed in the global reference system using the rotation matrix:

$$\mathbf{u_g} = \mathbf{\Psi}_\psi \mathbf{\Psi}_\theta \mathbf{\Psi}_\phi \mathbf{u_b} = \mathbf{\Psi} \mathbf{u_b} \tag{6}$$

where,

$$\mathbf{\Psi}_\psi = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

$$\mathbf{\Psi}_\theta = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{8}$$

$$\mathbf{\Psi}_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \tag{9}$$

$$\mathbf{\Psi} = \begin{bmatrix} \cos\psi\cos\theta & \cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi & \sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta & \cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\phi\cos\theta \end{bmatrix} \tag{10}$$

The angular velocity tensor $\mathbf{\Omega}$ of the body-mounted frame, expressed in the local coordinates of the rear frame assembly, can be written as follows:

$$\mathbf{\Omega} = \mathbf{\Psi}^\top \dot{\mathbf{\Psi}} = \begin{bmatrix} 0 & -\omega_z^B & \omega_y^B \\ \omega_z^B & 0 & -\omega_x^B \\ -\omega_y^B & \omega_x^B & 0 \end{bmatrix} \tag{11}$$

The relationship among the angular rates of the rear assembly $(\omega_x^B, \omega_y^B, \omega_z^B)$ and the time derivative of the roll ($\dot{\phi}$), pitch ($\dot{\theta}$), and yaw ($\dot{\psi}$) angles can be expressed as follows:

$$\dot{\phi} = \left(\omega_y^B \sin\phi + \omega_z^B \cos\phi\right) \tan(\theta) + \omega_x^B \tag{12}$$

$$\dot{\theta} = \omega_y^B \cos\phi - \omega_z^B \sin\phi \tag{13}$$

$$\dot{\psi} = \frac{\omega_y^B \sin\phi + \omega_z^B \cos\phi}{\cos\theta} \tag{14}$$

Revisions

Revision 2, July 29th 2016. The modifications with respect to the first revision consist in the addition of the equations to compute the coordinates of point P_2, and the addition of the "Notes" section.