IFToMM Benchmark Problem
Wheel on Tipping Table

Problem Description

The system considered is made up of two bodies, a rigid wheel that rests on a rigid table, where the table is held in place by a revolute joint at its mass centre. The table has one degree of freedom, which is a rotation around a horizontal axis. The wheel also has one degree of freedom; it can roll without slip on the table. To simulate the no-slip condition, it is constrained to the table in both directions at the point of contact, which is initially the mass centre of the table. However, the point of contact is allowed to move as the wheel rolls. The surface on which the wheel rolls is also assumed to pass through the table's centre of mass. A schematic diagram is shown in Figure 1.1.

Both bodies are assumed to have equal mass, in this case chosen as m = 1 kg, the wheel has a radius r = 0.25 m and the table is treated a thin square plate with side length l = 1 m. The inertias are calculated using standard expressions for thin laminar bodies.

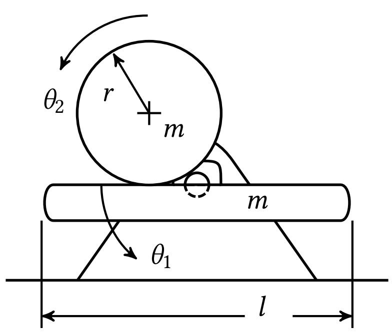

Figure 1.1: Wheel on tipping table.

Analysis

The equations of motion of the system can be generated using Lagrange's equations. The result is two coupled second-order non-linear ordinary differential equations, in the rotational coordinates of each body. The table orientation is defined by θ_1 and the wheel by θ_2. Both are absolute coordinates, i.e., measured relative to ground, and are zero when the table is level, and the point of contact of the wheel is at the table centre point.

$$T = \frac{1}{24} m l^2 \dot{\theta}_1^2 + \frac{1}{4} m r^2 \dot{\theta}_2^2 + \frac{1}{2} m r^2 ((\theta_2 - \theta_1)^2 \dot{\theta}_1^2 + \dot{\theta}_2^2) \tag{1.1}$$

$$V = mgr((\theta_1 - \theta_2)\sin\theta_1 + \cos\theta_1) \tag{1.2}$$

$$\left(\frac{1}{12} m l^2 + m r^2 (\theta_2 - \theta_1)^2\right) \ddot{\theta}_1 + m r(\theta_2 - \theta_1)\left(r(2\dot{\theta}_1\dot{\theta}_2 - \dot{\theta}_1^2) - g\cos\theta_1\right) = 0 \tag{1.3}$$

$$\frac{3}{2} m r^2 \ddot{\theta}_2 - m r^2 \dot{\theta}_1^2 (\theta_2 - \theta_1) - mgr\sin\theta_1 = 0 \tag{1.4}$$

These equations are linearized by discarding any products of variables, and substituting the small angle approximations. They are combined into a single vector equation.

$$\begin{bmatrix} \frac{1}{12} m l^2 & 0 \\ 0 & \frac{3}{2} m r^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} mgr & -mgr \\ -mgr & 0 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1.5}$$

The solution of the linear equations is found by substituting an exponential solution, giving the eigenvalue problem:

$$\det \begin{bmatrix} \frac{1}{12} m l^2 s^2 + mgr & -mgr \\ -mgr & \frac{3}{2} m r^2 s^2 \end{bmatrix} = 0 \tag{1.6}$$

Results

$$s = \frac{\pm\sqrt{-2g\left(3r \pm \sqrt{2l^2 + 9r^2}\right)}}{l} \tag{1.7}$$

When m = 1 kg, r = 0.25 m, l = 1 m, and g = 9.81 m/s^2:

$$s = \pm 6.791341869219082i \tag{1.8}$$

or

$$s = \pm 4.085624112006405 \tag{1.9}$$

There are four eigenvalues, including a pair of equal and opposite imaginary values, indicating an oscillatory mode. The oscillatory root shows rotational motion of the table around its axis, with a coupling to both translation and rotation of the wheel. Additionally, two non-oscillatory roots exist, one of which is unstable. The unstable mode predicts the wheel rolling forward off an increasingly inclined table, as would be expected.

The results file should list the four eigenvalues. Error can be computed as the l^2-norm of the difference between the benchmark and the computed eigenvalues.